deflection is played by a function with certain "smoothness" properties ($y_0(x) \in H_0^1(\Omega)$). The condition $y_0 \in E_2$ corresponds to the case in which the deflection at time t_0 is described by a function in $L_2(\Omega)$.

REFERENCES

1. KRASOVSKII N.N. and SUBBOTIN A.I., Positional Differential Games, Nauka, Moscow, 1974.
2. OSIPOV YU.S., On the theory of differential games in systems with distributed parameters. Dokl. Akad. Nauk SSSR, 223, 6, 1975.
3. KRYAZHIMSKII A.V. and OSIPOV YU.S., On modelling of the control in a dynamic system. Izv. Akad. Nauk SSSR, Tekh. Kibernetika, 2, 1983.
4. OSIPOV YU.S. and KRYAZHIMSKII A.V., On the dynamic solution of operator equations. Dokl. Akad. Nauk SSSR, 269, 3, 1983.
5. MAKSIMOV V.I., Positional modelling of controls and initial functions for Volterra systems. Differents. Urav., 23, 4, 1987.
6. GLOWINSKI R., LIONS J.-L. and TRÉMOLIÈRES R., Analyse numérique des inéquations variationelles, Vols.1-2, Paris, Dunod, 1976.
7. EVANS L.S., Non-linear evolution equations in an arbitrary Banach space. Israel J. Math., 26, 1, 1977.
8. DUVAUT G. and LIONS J.L., Les inéquations en mécanique et en physique. Paris, Dunod, 1962.
9. SAMARSKII A.A., Theory of difference schemes, Nauka, Moscow, 1977.

Translated by D.L.

PMM U.S.S.R.,Vol.52,No.5,pp.585-592,1988
Printed in Great Britain
0021-8928/88 $10.00+0.00

MULTILAYER FLOWS OF AN INCOMPRESSIBLE LIQUID OVER AN UNEVEN BOTTOM UNDER THE ACTION OF SURFACE PRESSURE*

K.A. BEZHANOV

The plane problem of the shear flow of an ideal heavy incompressible stratified liquid of finite depth over an uneven bottom is studied. The liquid has a finite number of layers and the stratification at their boundaries is discontinuous. An exact non-linear integrodifferential equation is obtained describing the internal and surface waves generated by the irregularities of the bottom, and by surface pressure. The basic properties of the spectrum of the linear problem proved in /1/, which generalize the results of /2, 3/, are formulated. A solution of the linear problem is obtained in the form of a Fourier series in terms of the eigenfunctions corresponding to the integral Fredholm equation or of the equivalent boundary value problem. The case of resonant reinforcement of the corresponding mode is discussed for the mean stream velocities close to, but smaller than the critical velocity. A non-linear problem of a streamlined flow with the formation of an internal two-soliton wave is considered for the case in which the mean stream velocities are close to and larger than the critical velocity.

1. Derivation of the basic equations. We consider the plane, steady-state flow of an ideal heavy incompressible stratified liquid above an uneven bottom, in the case when a known pressure is applied to the free surface of the liquid. The x axis is directed along the horizontal level of the bottom, and the y axis is directed vertically upwards. A one-dimensional shear flow is specified as $x \to -\infty$, with stable discontinuous stratification. When a one-dimensional stratified flow is acted upon by a known surface pressure $p_0(x)$ and by the irregularities of the bottom $y_0(x)$, it generates a two-dimensional stratified flow, the functions $p_0(x)$ and $y_0(x)$ are assumed to be continuous and finite, and the segment $[-x_0, x_0]$ is their common carrier. The liquid consists of n layers, the density and tangential

*Prikl.Matem.Mekhan.,52,5,751-759,1988

component of the velocity vector have first-order discontinuities at their boundaries $y_k(x)$ $(k = 1, 2, \ldots, n)$, while the pressure and normal component of the velocity vector are continuous. Moreover, it is convenient to assume that the space above the free boundary is filled with a fictitious fluid of zero density and velocity. In what follows, we shall change to dimensionless variables taking the depth of the stream h as the unit of length, and the mean density ρ_0 and mean velocity of the one-dimensional stream /4/ as the units of density and velocity.

Using the framework of the formulation of the problem given in /4/, we can formulate the following boundary value problem for the perturbation in the ordinate of the stream line $w(x, \eta)$ in Eulerian-Lagrangian variables, where the Lagrangian coordinate η gives, as $x \to -\infty$ the distance between the unperturbed stream line and the horizontal bottom

$$\frac{\partial}{\partial\eta}\left(a^2(\eta)\left(\frac{\partial w}{\partial\eta} + F_1 w\right)\right) + a^2(\eta)\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x} + F_2 w\right) - \nu\rho_\eta(\eta)\, w = 0, \quad (x, \eta) \in T \tag{1.1}$$

$$a^2(1)\left(\frac{\partial w}{\partial\eta}(x, 1) + F_1 w(x, 1)\right) - \nu\rho(1)\, w(x, 1) = -p_0(x)$$

$$\left[a^2(\eta)\left(\frac{\partial w}{\partial\eta}(x, \eta) + F_1 w(x, \eta)\right) - \nu\rho(\eta)\, w(x, \eta)\right]_k(x) = 0$$

$$[w(x, \eta)]_k(x) = 0, \quad k = 1, 2, \ldots, n-1$$

$$w(x, 0) = y_0(x), \quad \lim_{x \to -\infty} w(x, \eta) = 0$$

$$a^2(\eta) = \rho(\eta) V^2(\eta), \quad \nu = \frac{gh}{c^2} = \frac{1}{\mathrm{Fr}^2}$$

Here $[f]_k(x)$ is the jump in the value of the function $f(x, \eta)$ on passing through the k-th boundary of separation, and $\rho(\eta)$ and $V(\eta)$ are the density and velocity of the one-dimensional flow with discontinuities of the first kind at the points $\eta = \eta_k$ $(k = 0, 1, 2, \ldots, n)$, $\eta_0 = 0$, $\eta_n = 1$ and satisfying the conditions

$$\rho'(\eta) \leqslant 0, \quad \rho(\eta) \geqslant \rho_1 > (0), \quad V(\eta) \geqslant V_1 > 0$$

where g is the acceleration due to gravity, Fr is the Froude number, $F_1 w$ and $F_2 w$ are non-linear operators

$$F_1 w = -\frac{w_x^2 + 3w_\eta^2 + 2w_\eta^3}{2(1 + w_\eta)^2}, \quad F_2 w = -\frac{w_x w_\eta}{1 + w_\eta} \tag{1.2}$$

$$T = \bigcup_{k=1}^{n} T_k, \quad T_k = \{(x, \eta) : -\infty < x < +\infty, \eta_{k-1} < \eta < \eta_k\}$$

while the subscripts x and η indicate differentiation with respect to the corresponding variables.

We can reduce the boundary value problem (1.1) to the following non-linear integro-differential equation (IDE) with continuous and symmetrizable kernels:

$$w(x, \eta) - \nu\int_0^1 G(\eta, \xi)\, w(x, \xi)\, d\mu(\xi) = y_0(x) - q(\eta)\, p_0(x) - \int_0^\eta F_1 w\, d\xi + \int_0^1 a^2(\xi)\, G(\eta, \xi)\left(\frac{\partial^2 w}{\partial x^2}(x, \xi) + \frac{\partial}{\partial x} F_2 w\right) d\xi \tag{1.3}$$

where $d\mu(\eta)$ is the Lebesgue-Stieltjes measure generated by the monotonic function $\mu(\eta) = \rho(0) - \rho(\eta)$ and $G(\eta, \xi)$ is Green's function

$$G(\eta, \xi) = \begin{cases} q(\eta), & 0 \leqslant \eta \leqslant \xi \\ q(\xi), & \xi \leqslant \eta \leqslant 1 \end{cases}, \quad q(\eta) = \int_0^\eta \frac{d\tau}{a^2(\tau)} \tag{1.4}$$

which is a kernel of the integral Fredholm equation

$$\varphi(\eta) - \nu\int_0^1 G(\eta, \xi)\, \varphi(\xi)\, d\mu(\xi) = 0 \tag{1.5}$$

The integral Eq. (1.5) has, on the segment $\Omega = \{\eta\colon 0 \leqslant \eta \leqslant 1\}$, a denumerable set of simple positive real eigenvalues ν and an orthonormed system of eigenfunctions $\{\varphi_m(\eta)\}$ complete in $L_2(\mu, \Omega)$, and the resolvent is a meromorphic function of the parameter ν with simple poles at the points $\nu = \nu_m$

$$\Gamma(\xi, \eta, \nu) = \sum_{m=1}^{\infty} \frac{\varphi_m(\xi)\varphi_m(\eta)}{\nu - \nu_m} \tag{1.6}$$

From now on, the eigenvalues will be numbered in increasing order. They will also be called the critical values of the parameter ν and will have the corresponding critical velocities of propagation of long waves $c_m = \sqrt{gh/V_m}$.

Differentiating Eq.(1.5) with $\eta \neq \eta_k$ and using the expression for the derivative of $G(\eta, \xi)$ from (1.4), we can obtain the formula

$$\nu_m \int_\eta^1 \varphi_m(\xi)\, d\mu(\xi) = a^2(\eta)\varphi_m'(\eta) \tag{1.7}$$

Here and henceforth a prime will denote differentiation.

If we now invert the Fredholm operator appearing on the left-hand side of Eq.(1.3) and use formula (1.7), we can obtain the basic non-linear IDE describing the internal and surface waves generated by the irregularities of the bottom and by the surface pressure

$$w(x, \eta) + \int_0^1 a^2(\xi)\Gamma(\xi, \eta, \nu)\frac{\partial^2 w}{\partial x^2}(x, \xi)\, d\xi + \tag{1.8}$$

$$\int_0^\eta F_1 w\, d\xi + \int_0^1 a^2(\xi)\Gamma(\xi, \eta, \nu)\frac{\partial}{\partial x} F_2 w\, d\xi -$$

$$\nu \int_0^1 a^2(\xi)\Gamma_1(\xi, \eta, \nu) F_1 w\, d\xi = F(x, \eta)$$

$$F(x, \eta) = \chi(\eta, \nu) y_0(x) - \varkappa(\eta, \nu) p_0(x) \tag{1.9}$$

$$\varkappa(\eta, \nu) = 1 - \nu \int_0^1 \Gamma(\xi, \eta, \nu)\, d\mu(\xi)$$

$$\varkappa(\eta, \nu) = q(\eta) - \nu \int_0^1 \Gamma(\xi, \eta, \nu) q(\xi)\, d\mu(\xi)$$

$$\Gamma_1(\xi, \eta, \nu) = \sum_{m=1}^{\infty} \frac{\varphi_m'(\xi)\varphi_m(\eta)}{\nu_m(\nu - \nu_m)}$$

The essential difference between the IDE (1.8) and (1.3) is the fact that the homogeneous equation corresponding to (1.8) does not contain the Lebesgue-Stieltjes measure. It would be interesting to solve Eq.(1.8) numerically and compare the result with the results of the linear and non-linear theory obtained below.

2. Study of the linear problem. In order to solve the linear IDE corresponding to Eq.(1.8), we must consider the following integral Fredholm equation with continuous symmetric kernel /1, 4/:

$$z(\eta, \nu) - \lambda \int_0^1 a^2(\xi)\Gamma(\xi, \eta, \nu) z(\xi, \nu)\, d\xi = 0 \tag{2.1}$$

We can formulate for Eq.(2.1) an equivalent boundary value problem of the Sturm-Liouville problem type, containing two parameters λ and ν, which can be obtained from the boundary value problem (1.1), if we seek the solution of the linear problem of free waves using the method of separation of variables

$$\frac{d}{d\eta}\left(a^2(\eta)\frac{dz}{d\eta}\right) - (\lambda a^2(\eta) + \nu\rho'(\eta)) z = 0, \quad \eta \in \Omega_1 \tag{2.2}$$

$$a^2(1)\frac{dz}{d\eta}(1, \nu) - \nu\rho(1) z(1, \nu) = 0$$

$$\left[a^2(\eta)\frac{dz}{d\eta} - \nu\rho(\eta) z\right]_k = 0, \quad [z]_k = 0$$

$$z(0, \nu) = 0, \quad k = 1, \ldots, n-1$$

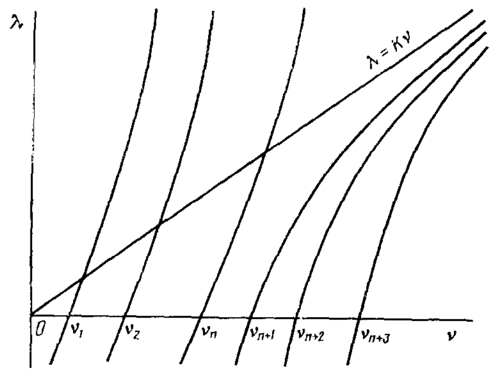
$$\Omega_1 = \{\eta\colon 0 < \eta < 1,\ \eta \neq \eta_k,\ k = 1, \ldots, n-1\}$$

The following properties of the eigenvalues and eigenfunctions were proved in /1, 4, 5/. Most of them, applied to the single-layer model and multilayer model without shear flows, were known earlier /2, 3, 6, 7/.

a) All eigenvalues $\lambda_m(\nu)$ of the integral Eq.(2.1) for the boundary value problem (2.2) are simple and real for $\nu \neq \nu_l$.

b) If $\nu_1 < \nu_2 < \ldots < \nu_l < \ldots$ are the critical values of the parameter ν, then for $\nu < \nu_1$ all eigenvalues $\lambda_m(\nu)$ will be negative.

c) The functions $\lambda_m(\nu)$ and $\lambda_m(\nu)/\nu$, $\nu \in (0, +\infty)$ are strictly monotonic increasing functions of the parameter ν.



d) If we arrange the numbers

$$\frac{1}{V^4(1)}, \quad \left(\frac{\rho(\eta_k+0)-\rho(\eta_k-0)}{a^2(\eta_k+0)+a^2(\eta_k-0)}\right)^2, \quad k=1, \ldots, n-1$$

in decreasing order and denote them by $\alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n$, then the following asymptotic formulas will hold as $\nu \to +\infty$:

$$\lambda_m(\nu) \sim \alpha_m \nu^2, \quad m = 1, \ldots, n$$
$$\lambda_m(\nu) \sim K\nu, \quad m = n+1, n+2, \ldots; \quad K = \max_{\eta \in \Omega} \frac{N(\eta)}{V^2(\eta)}$$

where $N(\eta)$ is the Brunt-Vaisala frequency.

e) For $\nu \in (\nu_l, \nu_{l+1})$ the characteristic numbers $\lambda_1(\nu), \ldots, \lambda_l(\nu)$ are positive, and all the remaining ones are negative.

f) The integral Eq.(2.1) or the equivalent boundary value problem (2.2), has a system of eigenfunctions complete in $L_2(\Omega)$ and orthonormed with weight $a^2(\eta)$.

g) The eigenfunction $z_m(\eta, \nu)$ has exactly $m-1$ node-type zeros in the interval $\eta \in (0, 1)$, and the zeros of the functions $z_m(\eta, \nu)$ and $z_{m+1}(\eta, \nu)$ alternate in the same interval. When $\eta \in \Omega$, the eigenfunction $z_m(\eta, \nu)$ vanishes m times since $z_m(0, \nu) = 0$.

h) If the Brunt-Vaisala frequency has a unique maximum at the point η_0, then a sufficiently large ν_0 can be found in any of its ε-neighbourhood such that when $\nu > \nu_0$, all $n-1$ zeros of the eigenfunction $z_m(\eta, \nu)$ will be concentrated in the segment $[\eta_0 - \varepsilon, \eta_0 + \varepsilon]$ and the internal waves will be captured within a thermal wedge.

Properties a) - e) enable us to draw the dispersion curves shown in the figure. Thus we can have, in a stream of a discontinuously stratified liquid, when there are shear flows, n wave modes corresponding to the free surface and separation boundaries, with dispersion curves intersecting the straight line $\lambda = K\nu$, and an infinite number of wave modes corresponding to a continuous stratification within each layer, with dispersion curves lying below the straight line $\lambda = K\nu$.

The properties of the spectrum of the linear problem formulated here enable us to solve the linear IDE corresponding to Eq.(1.8), using the Fourier method. If we expand $w(x, \eta)$, $\chi(\eta, \nu)$ and $\varkappa(\eta, \nu)$ in a Fourier series over the system $\{z_m(\eta, \nu)\}$ complete in $L_2(\Omega)$ and substitute it into the linear IDE, we obtain the following expressions for the unknown Fourier coefficients $w_m(x)$:

$$\frac{d^2 w_m}{dx^2} + \lambda_m w_m = \lambda_m f_m(x), \quad |x| < \infty \tag{2.3}$$

$$f_m(x) = \chi_m y_0(x) - \varkappa_m p_0(x)$$

$$\chi_m = \int_0^1 a^2(\xi) z_m(\xi, \nu)\, d\xi - \frac{\nu}{\lambda_m} \int_0^1 z_m(\xi, \nu)\, d\mu(\xi)$$

$$\varkappa_m = \int_0^1 a^2(\xi) q(\xi) z_m(\xi, \nu)\, d\xi - \frac{\nu}{\lambda_m} \int_0^1 q(\xi) z_m(\xi, \nu)\, d\mu(\xi)$$

Here χ_m and $\varkappa_m$ are the Fourier coefficients of the functions $\chi(\eta, \nu)$ and $\varkappa(\eta, \nu)$.

The solution of (2.3), which vanishes together with its derivative as $x \to -\infty$, has the form

$$w_m(x) = w_m^+(x) = \sqrt{\lambda_m} \int_{-x_0}^{x} f_m(\xi) \sin \sqrt{\lambda_m}(x-\xi)\, d\xi, \quad \lambda_m > 0 \tag{2.4}$$

$$w_m(x) = w_m^-(x) = \frac{\sqrt{|\lambda_m|}}{2} \int_{-x_0}^{x_0} f_m(\xi) \exp(-\sqrt{|\lambda_m|}\,|x-\xi|)\, d\xi, \tag{2.5}$$

$$\lambda_m < 0$$

and finally

$$w(x, \eta) = \sum_{m=1}^{L} w_m^+(x) z_m(\eta, \nu) + \sum_{m=L+1}^{\infty} w_m^-(x) z_m(\eta, \nu) \tag{2.6}$$

Thus when $\nu \in (\nu_l, \nu_{l+1})$, the solution of the linear problem (2.4)-(2.6) is given in the form of a sum $l = L(\nu)$ of harmonic waves, and an infinite number of exponentially decaying perturbations /8, 9/.

In the linear case in question we find that the l-th mode is resonantly amplified as $\nu \to \nu_l + 0$ $(c \to c_l - 0)$. In this case the solution simplifies noticeably and reduces to a single term of the series (2.6). The analyticity of the functions $\lambda_l(\nu)$ and $z_l(\eta, \nu)$ with respect to the variable ν in the neighbourhood of the point $\nu = \nu_l$, was proved in /1/. Therefore we find, taking into account the fact that $\lambda_l(\nu_l) = 0$, that the following asymptotic formulas hold:

$$\lambda_l(\nu) = \frac{\nu - \nu_l}{A_{ll}} + o(\nu - \nu_l), \quad A_{ll} = \int_0^1 a^2(\xi)\varphi_l^2(\xi)\, d\xi \tag{2.7}$$

$$z_l(\eta, \nu) = \frac{\varphi_l(\eta)}{\sqrt{A_{ll}}} + O(\nu - \nu_l)$$

Moreover, formulas (1.6), (1.7) and (2.3) yield the following approximate formulas as $\nu \to \nu_l + 0$:

$$\Gamma(\xi, \eta, \nu) \approx \frac{\varphi_l(\xi)\varphi_l(\theta)}{\nu - \nu_l} \tag{2.8}$$

$$\chi_l \approx \frac{\nu_l}{(\nu_l - \nu)\sqrt{A_{ll}}} \int_0^1 \varphi_l(\xi)\, d\mu(\xi) = \frac{\chi_0}{(\nu_l - \nu)\sqrt{A_{ll}}}$$

$$\varkappa_l \approx \frac{\nu_l}{(\nu_l - \nu)\sqrt{A_{ll}}} \int_0^1 q(\xi)\varphi_l(\xi)\, d\mu(\xi) = \frac{\varkappa_0}{(\nu_l - \nu)\sqrt{A_{ll}}}$$

$$\varkappa_0 = a^2(0)\varphi_l'(0), \quad \varkappa_0 = \varphi_l(1)$$

where the integral in the formula for $\varkappa_l$ is calculated by substituting into it the expression for $q(\xi)$ from (1.4), changing the order of integration, and using formula (1.7). Then, if

$$P = \int_{-x_0}^{x_0} p_0(x)\, dx = o(\sqrt{\nu - \nu_l})), \quad S = \int_{-x_0}^{x_0} y_0(x)\, dx = o(\sqrt{\nu - \nu_l}) \tag{2.9}$$

i.e. when the quantities P and S are specifically coordinated with the degree of approximation of the mean stream velocity to the critical velocity of propagation of the long wave, then the assumptions of the linear theory become valid and formulas (2.4)-(2.9) yield, when $x > x_0$,

$$w(x, \eta) = \frac{\varkappa_0 P - \chi_0 S}{\sqrt{(\nu - \nu_l)A_{ll}}} \sin \sqrt{\lambda_l}\, x\varphi_l(\eta) + O(\delta)$$

where $O(\delta)$ is a quantity of the order of P and S.

3. Derivation of the non-linear equation as $\nu \to \nu_l - 0$.

Let the mean velocity of a one-dimensional flow be close to one of the critical velocities of propagation of long waves and exceeding it: $c \to c_l + 0$ $(\nu \to \nu_l - 0)$. In this case the linear theory can no longer be used and the approximate non-linear theory yields a qualitatively novel effect illustrated by the appearance of a "precursor" in the form of solitons situated along the vertical in front of the streamlined perturbation sources, and of the solitons also distributed along the vertical, but behind the streamlined sources.

When $\nu \to \nu_l - 0$, the asymptotic formulas (2.8), in which we put $A_{ll} = 1$ hold, and we also have

$$\Gamma_1(\xi, \eta, \nu) \approx -\frac{\varphi_l'(\xi)\varphi_l(\eta)}{\nu_l \varepsilon^2}, \quad \varepsilon^2 = \nu_l - \nu \tag{3.1}$$

Substituting formulas (2.8) and (3.1) into the IDE (1.8), we obtain

$$\varepsilon^2 \frac{w(x,\eta)}{\varphi_l(\eta)} - \int_0^1 a^2(\xi)\varphi_l(\xi)\frac{\partial^2 w}{\partial x^2}(x,\xi)\,d\xi - \int_0^1 a^2(\xi)\varphi_l(\xi) \times \tag{3.2}$$

$$\frac{\partial}{\partial x} F_2 w(x,\xi)\,d\xi + \int_0^1 a^2(\xi)\varphi_l'(\xi) F_1 w(x,\xi)\,d\xi = \chi_0 y_0(x) - \varkappa_0 p_0(x)$$

From (3.2) it follows that we can write, to a first approximation,

$$w(x,\eta) \approx u(x)\varphi_l(\eta) \tag{3.3}$$

Moreover we can write the approximate formulas for $F_1 w$ and $F_2 w$ in the form

$$F_1 w \approx -\frac{3}{2} w_\eta^2 + 2w_\eta^3 - \frac{1}{2} w_x^2, \quad F_2 w \approx -w_x w_\eta \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2), we obtain the following non-linear differential equations:

$$\frac{1}{b^2}\frac{d^2u}{dx^2} + \frac{3}{2d}u^2 - \frac{2}{e}u^3 - \varepsilon^2 u - Eu\frac{du}{dx} + \tag{3.5}$$

$$\frac{E}{2}\left(\frac{du}{dx}\right)^2 = \varkappa_0 p_0(x) - \chi_0 y_0(x)$$

$$\frac{1}{b^2} = \int_0^1 a^2(\xi)\varphi_l^2(\xi)\,d\xi, \quad \frac{1}{d} = \int_0^1 a^2(\xi)(\varphi_l'(\xi))^3\,d\xi \tag{3.6}$$

$$\frac{1}{e} = \int_0^1 a^2(\xi)(\varphi_l'(\xi))^4\,d\xi, \quad E = \int_0^1 a^2(\xi)\varphi_l'(\xi)\varphi_l^2(\xi)\,d\xi$$

Below we assume that the integral in the expression for the constant d in formulas (3.6) does not vanish. The study of the degenerate case $((1/d) = 0)$ poses great technical difficulties and is of no great interest.

4. Study of the non-linear problem. The non-linear Eq. (3.5) is still sufficiently complicated to make its analytic investigation difficult; therefore we shall construct its approximate solution below. To do this we divide the x axis into three intervals: $(-\infty, -x_0]$, $(-x_0, x_0)$ and $[x_0, +\infty)$. The solution of (3.5) becomes simpler for each of these intervals, and the solutions will merge at the points $\pm x_0$ by virtue of the suitable choice of arbitrary constants.

When $|x| \geqslant x_0$, we seek a long-wave approximation of Eq. (3.5). This will become homogeneous since the functions $p_0(x)$ and $y_0(x)$ are finite. We expand the independent variable in terms of the small parameter ε, and seek the solution in the form of a series in powers of ε, restricting ourselves to the first term of the expansion

$$\tau = \varepsilon bx, \quad u = \varepsilon^2 dv(\tau) + \ldots$$

This yields the stationary Korteweg-de Vries equation

$$u''(\tau) + \frac{3}{2}v^2(\tau) - v(\tau) = 0$$

which has a two-parameter family of periodic solutions in the form of a knoidal wave /11/. Finally we have

$$u(x) = \begin{cases} u_-(x), & x \leqslant -x_0 \\ u_+(x), & x \geqslant x_0 \end{cases} \tag{4.1}$$

$$u_\mp(x) = \varepsilon^2 d\left(\alpha + (\beta - \alpha)\,\mathrm{cn}^2\left(\frac{1}{2}\varepsilon\delta b(x \pm x_0) \pm A_\mp, s\right)\right)$$

$$\beta = (1 - \alpha + \delta)/2, \quad \delta = \sqrt{(1-\alpha)(1+3\alpha)}$$

$$\gamma = (\alpha - 1 + \delta)/2, \quad s^2 = (\beta - \alpha)/(\beta + \alpha)$$

where α, A_- and A_+ are arbitrary positive constants.

When $\alpha \to 0$, formulas (4.1) yield $\beta \to 1, \gamma \to 0, \delta \to 1, s \to 1$, and the knoidal wave degenerates into a soliton. In this case

$$u_{\mp}(x) = \varepsilon^2 d \operatorname{sech}^2\left(\frac{1}{2}\varepsilon b(x \pm x_0) \pm A_{\mp}\right) \quad (4.2)$$

The arbitrary constant A_- and A_+ are obtained after finding the approximate solution of Eq.(3.5) on the segment $[-x_0, x_0]$ and matching the solutions and their derivatives at $x = \pm x_0$. Using formulas (4.2) we obtain

$$u(\mp x_0) = \varepsilon^2 d(1 - B_{\mp}^2), \ B_{\pm} = \operatorname{th} A_{\pm}$$
$$u'(\mp x_0) = \mp \varepsilon^3 b d B_{\mp}(1 - B_{\mp}^2) \quad (4.3)$$

When $x \in (-x_0, x_0)$, we obtain the solution of (3.5) assuming that

$$x_0 = O(1), \ S = O(\varepsilon^3), \ P = O(\varepsilon^3)$$
$$u(x) = O(\varepsilon^2), \ u'(x) = O(\varepsilon^3), \ u''(x) = O(\varepsilon^3)$$

In this case Eq.(3.5) simplifies considerably and takes the form

$$\frac{d^2u}{dx^2} = b^2(\varkappa_0 p_0(x) - \chi_0 y_0(x)) \quad (4.4)$$

Solving the Cauchy problem for Eq.(4.4) with initial conditions (4.3) at the point $-x_0$, we obtain

$$\frac{du}{dx} = b^2 \int_{-x_0}^{x} (\varkappa_0 p_0(\xi) - \chi_0 y_0(\xi))\, d\xi - \varepsilon^3 b\, dB_-(1 - B_-^2) \quad (4.5)$$

$$u(x) = b^2 \int_{-x_0}^{x} (x - \xi)(\varkappa_0 p_0(\xi) - \chi_0 y_0(\xi))\, d\xi +$$
$$\varepsilon^2 d(1 - B_-^2) - \varepsilon^3 b d B_-(1 - B_-^2)(x + x_0)$$

Matching the solutions (4.2) and (4.5) at the point x_0 we obtain an algebraic system for determining the constants B_- and B_+. From (4.3) and (4.5) we obtain

$$B_-(1 - B_-^2) = \frac{b}{2d\varepsilon^3}(\varkappa_0 P - \chi_0 S), \quad A_- = A_+ \quad (4.6)$$

When $0 \leqslant B_{\pm} \leqslant 1$, system (4.6) has two solutions which satisfy the inequalities

$$0 \leqslant \frac{b}{d\varepsilon^3}(\varkappa_0 P - \chi_0 S) \leqslant \frac{4}{3\sqrt{3}} \quad (4.7)$$
$$0 \leqslant B_-^{(1)} \leqslant (1/\sqrt{3}) \leqslant B_-^{(2)} \leqslant 1$$

i.e. when the parameters of the incoming flow and ε are all given, then the two-soliton solution may exist only when the quantities P and S are related to each other in a prescribed manner, and out of the two solutions we must choose one which is physically realizable.

For fixed ε and $P \to 0$, $S \to 0$, the first solution degenerates into a one-dimensional flow in which case the solitons diverge and recede to infinity, and the second solution degenerates into a unified wave since the solitons converge and merge with each other. Since the one-dimensional flow is unstable when $\nu \to \nu_l - 0$ while the unified wave is stable, we can assume that the second equation holds /12/. It would be of interest to test the non-stationary problem for stability, and to investigate the problems of the existence of the approximate solutions obtained.

From the formulas (4.2) and (4.6) it follows that when the parameters of one-dimensional flow are given, two solitons with apices at the points $\pm(x_0 + 2A/b\varepsilon)$ will be distributed symmetrically about the origin of coordinates, with their distribution depending on the quantities P, S and ε, while the amplitudes will depend only on ε.

Fusing the solution (4.5) with the solution for the knoidal wave (4.1) and the soliton solution (4.2) at $x \leqslant -x_0$ with solution (4.5) and the solution for the knoidal wave (4.1) at $x \geqslant x_0$, we obtain a system of two equations for determining α, A_- and A_+, i.e. we obtain a one-parameter family of solutions which, as $\alpha \to 0$, transforms into the two-soliton solution obtained above.

5. One-layer and two-layer models. In the case of a one-layer model $\rho(\eta) = V(\eta) = 1$, and we obtain, from the boundary value problem (2.2), the corresponding normed eigenfunction

$$\varphi_1(\eta) = \eta, \ \eta \in \Omega \quad (5.1)$$

The condition (4.8) takes the form

$$0 \leqslant P - S \leqslant 4\varepsilon^3/9 \quad (5.2)$$

and from this it follows that for a two-soliton solution to exist it is necessary that the total pressure on the free surface should be greater than S. For example, where there is no surface pressure we find that in the case of obstruction condition (5.2) implies that in a homogeneous liquid there is no two-soliton solution.

In the case of a two-layer model, we consider the case of the piecewise-constant density distribution given in /4/. When $\nu \to \nu_1 - 0$, the normed eigenfunction $\varphi_1(\eta)$ has the form (5.1) and condition (4.7) is identical with (5.2), while when $\nu \to \nu_2 - 0$, we have

$$\varphi_2(\eta) = \frac{1}{h_1 h_2 \sqrt{\Delta\rho}} \begin{cases} h_2\eta, & 0 \leqslant \eta \leqslant h_1 \\ h_1(1-\eta), & h_1 \leqslant \eta \leqslant 1 \end{cases}$$

where $\Delta\rho > 0$ is a small density change and $h_1 + h_2 = 1$. Condition (4.7) for a two-soliton solution to exist takes the form

$$0 \leqslant (h_1 - h_2)S \leqslant 4\varepsilon^3/(9h_1^3h_2^2(\Delta\rho)^{3/2})$$

and is satisfied for some ε and $\Delta\rho$ for the case when the upper layer is not thicker than the bottom layer irrespective of the value of the surface tension.

REFERENCES

1. BEZHANOV K.A. and TER-KRIKOROV A.M., Study of the spectral problem of the theory of stratified flows of an ideal incompressible heavy fluid. Differents. Urav. 23, 11, 1987.
2. YANOWITCH M., Gravity waves in a heterogeneous incompressible fluid, Communs. Pure and Appl. Math. 15, 1, 1962.
3. DOTSENKO S.F., On the structure of wave motion in a stream when the density changes arbitrarily with depth. Morskiye Gidrifizich. Issledovaniya, Sevastopol, Izd. MGI Akad. Nauk UKrSSR, 3, 1973.
4. BEZHANOV K.A. and TER-KRIKOROV A.M., Multilayer stationary flows of an ideal incompressible fluid over an uneven floor, PMM, 48, 5, 1984.
5. BEZHANOV K.A. and TER-KRIKOROV A.M., Flow of a multilayer incompressible heavy fluid past a body. PMM, 49, 3, 1985.
6. LONG R.R., Some aspects of the flow of stratified fluids. I. A theoretical investigation, Tellus. 5, 1, 1953.
7. YIH C.S., Dynamics of Non-homogeneous Fluids. New York, MacMillan, 1965.
8. DOTSENKO S.F. and CHERKESOV L.V., On the effect of a continuous change in the density of a fluid on the waves generated by moving surface pressures. Izv. Akad. Nauk SSSR, MZhG, 6, 1973.
9. CHERKESOV L.V., Hydrodynamics of Surface and Internal Waves. Naukova Dumka, Kiev, 1976.
10. TER-KRIKOROV A.M., On internal waves in an inhomogeneous liquid, PMM, 26, 6, 1962.
11. STETENSKII L.N., Theory of Wave Motions of a Liquid, Nauka, Moscow, 1977.
12. KAMENKOVICH V.M. and MONIN A.S., Eds., Physics of the Ocean. 2, Nauka, Moscow, 1978.

Translated by L.K.